

# REMOVABLE SETS FOR ORLICZ-SOBOLEV SPACES

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**ABSTRACT.** We study removable sets for the Orlicz-Sobolev space  $W^{1,\Psi}$ , for functions of the form  $\Psi(t) = t^p \log^\lambda(e + t)$ . We show that  $(p, \lambda)$ -porous sets lying in a hyperplane are removable and that this result is essentially sharp.

## 1. INTRODUCTION

In this paper, we consider removability problems for Orlicz-Sobolev spaces  $W^{1,\Psi}$  with  $\Psi(t) = t^p \log^\lambda(e + t)$ . We generalize results of Koskela in [Kos99] for the usual Sobolev spaces. Let us first recall some definitions. Let  $\Omega$  be an open set in  $\mathbb{R}^n$ ,  $n \geq 2$ . We say that  $u$  is in the Sobolev space  $W^{1,p}(\Omega)$  if  $u \in L^p(\Omega)$ ,  $1 \leq p < \infty$ , and there are functions  $\partial_j u \in L^p(\Omega)$ ,  $j = 1, \dots, n$ , so that

$$(1.1) \quad \int_{\Omega} u \partial_j \phi \, dx = - \int_{\Omega} \phi \partial_j u \, dx$$

for each test function  $\phi \in C_0^1(\Omega)$  and all  $1 \leq j \leq n$ . If  $E \subset \mathbb{R}^n$  is a closed set of zero Lebesgue  $n$ -measure, then we say that  $E$  is removable for  $W^{1,p}$  if  $W^{1,p}(\mathbb{R}^n \setminus E) = W^{1,p}(\mathbb{R}^n)$  as sets. It is not hard to check that  $E$  is removable if and only if the functions  $\partial_j u \in L^p(\mathbb{R}^n \setminus E)$  satisfy (1.1) (with  $\Omega = \mathbb{R}^n$ ) for each  $\phi \in C_0^1(\mathbb{R}^n)$  and not only for  $\phi \in C_0^1(\mathbb{R}^n \setminus E)$ . Similarly to the definition of  $W^{1,p}(\Omega)$ ,  $W^{1,\Psi}(\Omega)$  refers to the class of functions in  $L^\Psi(\Omega)$  with  $\partial_j u \in L^\Psi(\Omega)$ ,  $j = 1, 2, \dots, n$ .

**Definition 1.1.** If  $E \subset \mathbb{R}^n$  is a closed set of zero Lebesgue  $n$ -measure, then we say that  $E$  is removable for  $W^{1,\Psi}$  if  $W^{1,\Psi}(\mathbb{R}^n \setminus E) = W^{1,\Psi}(\mathbb{R}^n)$  as sets.

It is easy to see that removability is a local question as in the classical case. That is,  $E$  is removable for  $W^{1,\Psi}$  if and only if for each  $x \in E$  there is  $r > 0$  so that  $W^{1,\Psi}(B(x, r) \setminus E) = W^{1,\Psi}(B(x, r))$  as sets. Moreover, if  $E \subset \Omega$  for some open set  $\Omega$ , then

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$E$  is removable for  $W^{1,\Psi}$  if and only if  $W^{1,\Psi}(\Omega \setminus E) = W^{1,\Psi}(\Omega)$  as sets. Observe that, to verify the removability, it is enough to consider the functions  $u \in C^1(\Omega \setminus E) \cap W^{1,\Psi}(\Omega \setminus E)$  as  $W^{1,\Psi}$  is Banach space and smooth functions are dense in  $W^{1,\Psi}(\Omega \setminus E)$  for a doubling function  $\Psi$ .

In this paper, we study the removability of compact sets  $E \subset \mathbb{R}^{n-1}$ . Given  $1 < p \leq n$ , Koskela showed in [Kos99] that there are compact sets  $E \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$  that are removable for  $W^{1,p}(\mathbb{R}^n)$ , but not for  $W^{1,q}(\mathbb{R}^n)$  for any  $q < p$ . This was done by introducing the class of  $p$ -porous sets. It is then natural to ask if a similar result holds for  $W^{1,\Psi}(\mathbb{R}^n)$ , for Orlicz functions  $\Psi(t) = t^p \log^\lambda(e + t)$  in terms of  $\lambda$ . We prove that this is indeed the case by studying a generalization of  $p$ -porosity, the  $(p, \lambda)$ -porosity defined in Section 4 below.

**Theorem A.** Let  $E \subset \mathbb{R}^{n-1}$  be compact. Let  $1 < p < n, \lambda \in \mathbb{R}$  or  $p = 1, \lambda > 0$  or  $p = n, \lambda \leq n - 1$ . If  $E$  is  $(p, \lambda)$ -porous, then  $E$  is removable for  $W^{1,\Psi}$  in  $\mathbb{R}^n$ , where  $\Psi(t) = t^p \log^\lambda(e + t)$ . Moreover, for each pair  $(p, \lambda)$  as above, there is a  $(p, \lambda)$ -porous set  $E \subset \mathbb{R}^{n-1}$  that is not removable for  $W^{1,\Psi'}$  for  $\Psi'(t) = t^p \log^{\lambda-\epsilon}(e + t)$  for any  $\epsilon > 0$ .

The restrictions  $\lambda > 0$  for  $p = 1$  and  $\lambda \leq n - 1$  for  $p = n$  are natural, see the discussion in Section 3 below.

The main idea behind the removability of  $(p, \lambda)$ -porous sets is the following. As mentioned above, it suffices to prove that (1.1) holds for each  $u \in C^1(\Omega \setminus E) \cap W^{1,\Psi}(\Omega \setminus E)$  and for each  $\phi \in C_0^1(\Omega)$ . By the Fubini theorem and the usual integration by parts it suffices to show that the one sided limits  $\lim_{t \rightarrow 0+} u(x', t)$  and  $\lim_{t \rightarrow 0-} u(x', t)$  coincide for  $H^{n-1}$ -a.e.  $x = (x', 0) \in E$ . This is established via sharp capacity estimates and the existence of “holes” in  $E$  guaranteed by the porosity condition. The same idea was used also in [Kos99], but the necessary estimates and even the definition of porosity is more novel in our setting.

Similarly to [Kos99], Theorem A yields the following result on Orlicz-Poincaré inequalities:

**Corollary.** Let  $n \geq 2$  be an integer,  $1 < p < n$  and  $\lambda \in \mathbb{R}$ . There is a locally compact  $n$ -regular metric space that supports an Orlicz  $(p, \lambda)$ -Poincaré inequality but does not support an Orlicz  $(p, \lambda - \epsilon)$ -Poincaré inequality for any  $\epsilon > 0$ .

The above corollary shows that there is no self-improvement in an Orlicz  $(p, \lambda)$ -Poincaré inequality in the non-complete setting (notice that  $\mathbb{R}^n \setminus E$  is not complete). This partially motivates this note. For a complete  $n$ -regular space, an Orlicz  $(p, \lambda)$ -Poincaré inequality,  $1 < p < \infty$ , always improves even in  $p$  when  $\lambda < p - 1$ . For the case  $\lambda = 0$ , see [KZ08] and for general  $\lambda$ , see [Dej].

For the definition of an Orlicz  $(p, \lambda)$ -Poincaré inequality see Section 2 below. The definition of porosity is given in Section 4.

In order to make this paper more readable, we organize it as follows. In Section 2 we recall definitions and preliminary results. As Theorem A admits a more elementary proof in the planar case, we begin by proving Theorem A in Section 3 in the plane. In Section 4 we describe the modifications necessary for handling the higher dimensional situation.

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## 2. NOTATION AND PRELIMINARIES

A function  $\Psi : [0, \infty) \rightarrow [0, \infty)$  is a Young function if

$$\Psi(s) = \int_0^s \psi(t) dt,$$

where  $\psi : [0, \infty) \rightarrow [0, \infty)$  with  $\psi(0) = 0$ , is an increasing, left-continuous function which is neither identically zero nor identically infinite on  $(0, \infty)$ . A Young function  $\Psi$  is convex, increasing, left-continuous and satisfies

$$\Psi(0) = 0, \lim_{t \rightarrow \infty} \Psi(t) = \infty.$$

The generalized inverse of a Young function  $\Psi$ ,  $\Psi^{-1} : [0, \infty] \rightarrow [0, \infty]$ , is defined by the formula

$$\Psi^{-1}(t) = \inf\{s : \Psi(s) > t\},$$

where  $\inf(\emptyset) = \infty$ . A Young function  $\Psi$  and its generalized inverse satisfy the double inequality

$$\Psi(\Psi^{-1}(t)) \leq t \leq \Psi^{-1}(\Psi(t))$$

for all  $t \geq 0$ . In this article we will only consider the Young functions  $\Psi(t) = t^p \log^\lambda(e+t)$ ,  $1 \leq p \leq n$ ,  $\lambda \in \mathbb{R}$ . For a general Young function  $\Psi$ , the Orlicz space  $L^\Psi(\Omega)$  is defined by

$$L^\Psi(\Omega) = \{u : \Omega \rightarrow [-\infty, \infty] : u \text{ measurable, } \int_\Omega \Psi(\alpha|u|) dx < \infty \text{ for some } \alpha > 0\}.$$

As in the theory of  $L^p$ -spaces, the elements in  $L^\Psi(\Omega)$  are actually equivalence classes consisting of functions that differ only on a set of measure zero. The Orlicz space  $L^\Psi(\Omega)$  is a vector space and, equipped with the Luxemburg norm

$$\|u\|_{L^\Psi(\Omega)} = \inf\{k > 0 : \int_\Omega \Psi\left(\frac{|u|}{k}\right) dx \leq 1\},$$

a Banach space, see [RR91, Theorem 3.3.10]. A function  $u \in L^\Psi(\Omega)$  is in the Orlicz-Sobolev space  $W^{1,\Psi}(\Omega)$  if its weak partial derivatives (distributional derivatives)  $\partial_j u$  belong to  $L^\Psi(\Omega)$  for all  $1 \leq j \leq n$ . The space  $W^{1,\Psi}(\Omega)$  is a Banach space with respect to the norm

$$\|u\|_{W^{1,\Psi}(\Omega)} = \|u\|_{L^\Psi(\Omega)} + \|\nabla u\|_{L^\Psi(\Omega)},$$

where  $\nabla u = (\partial_1 u, \dots, \partial_n u)$ . For a proof, see for example [RR91, Theorem 9.3.3]. For more about Young functions, Orlicz spaces and Orlicz-Sobolev spaces, see e.g. [Tuo04, RR91]. Recall that a Young function  $\Psi : [0, \infty) \rightarrow [0, \infty)$  is said to be doubling if there is a constant  $C > 0$ , called a doubling constant of  $\Psi$ , such that

$$\Psi(2t) \leq C\Psi(t)$$

for each  $t \geq 0$ . Sometimes the doubling condition is also called the  $\Delta_2$ -condition.

Let us also recall the Poincaré and the  $\Psi$ -Poincaré inequalities. A pair  $u \in L^1_{\text{loc}}(\Omega)$  and a measurable function  $g \geq 0$  satisfy a  $(1, p)$ -Poincaré inequality,  $p \geq 1$ , if there exist constants  $C_p > 0$  and  $\tau \geq 1$ , such that

$$(2.1) \quad \int_B |u - u_B| dx \leq C_p r \left( \int_{\tau B} g^p dx \right)^p$$

for each ball  $B = B(x, r)$  satisfying  $\tau B \subset \Omega$ . Recall that if  $\Omega \subset \mathbb{R}^n$  and  $u \in W^{1,1}_{\text{loc}}(\Omega)$ , then the inequality (2.1) holds for  $g = |\nabla u|$  with  $\tau = 1$ ,  $p = 1$  and the constant depending only on  $n$ . Similarly, a function  $u \in L^1_{\text{loc}}(\Omega)$  and a measurable function  $g \geq 0$  satisfy a  $\Psi$ -Poincaré inequality, if there exist constants  $C_\Psi > 0$  and  $\tau \geq 1$ , such that

$$(2.2) \quad \int_B |u - u_B| dx \leq C_\Psi r \Psi^{-1} \left( \int_{\tau B} \Psi(g) dx \right)$$

for each ball  $B = B(x, r)$  satisfying  $\tau B \subset \Omega$ . Here  $u_B$  is the average of  $u$  in  $B(x, r)$  and the barred integrals are the averaged integrals, that is  $\bar{f}_A v d\mu = \mu(A)^{-1} \int_A v d\mu$ .

### 3. THE PLANAR CASE

Let  $E \subset (0, 1)$  be a compact set in  $\mathbb{R} \subset \mathbb{R}^2$ . We say that  $E$  is  $(p, \lambda)$ -removable if  $E$  is removable for  $W^{1, \Psi}$  for the function  $\Psi(t) = t^p \log^\lambda(e + t)$ , where  $p \in [1, \infty)$  and  $\lambda$  is any real number. It is easy to check that  $(p, \lambda)$ -removability is equivalent to the requirement that for each  $u \in W^{1, \Psi}(B(0, 2) \setminus E) \cap C^1(B(0, 2) \setminus E)$ ,  $u^+(x) = u^-(x)$  holds for  $H^1$ -a.e.  $x \in E$ . Here  $u^+(x) = \lim_{t \rightarrow 0+} u(x_1, t)$ ,  $u^-(x) = \lim_{t \rightarrow 0-} u(x_1, t)$  and these limits exist for  $H^1$ -a.e.  $x = (x_1, 0) \in E$ , by the Fubini theorem and the fundamental theorem of calculus. Removability of a set  $E$  may depend on the exponents  $p$  and  $\lambda$ . Indeed, when  $p > 2$ ,  $\lambda \in \mathbb{R}$  and  $p = 2$ ,  $\lambda > 1$  the complementary intervals of  $E$  in  $(0, 1)$  play no role for the removability, since in this case any totally disconnected closed set  $E \subset (0, 1)$  is removable for  $W^{1, \Psi}$  (see [Kos99, prop.2.1], [RR91, sec.9.3] and [Ada77, sec.2]). The point here is that, for these values of  $p, \lambda$ , one has  $u^+(x) = u^-(x)$  for all  $x = (x_1, 0)$ .

The idea behind our definition of porosity and its applicability is the the following. If a continuous function  $u \in W^{1, \Psi}$  equals one on  $I_1$  and zero on  $I_2$  in Figure 1, then using a chaining argument and the usual Poincaré inequality one can verify the capacity type estimate

$$\int_{B(x, r)} \Psi(|\nabla u|) \geq c s^{2-p} \log^\lambda \left( \frac{1}{s} \right)$$

for  $1 \leq p < 2$ , where  $s = \text{diam}(I_2)$  and one has a similar estimate for  $p = 2$  also. On the

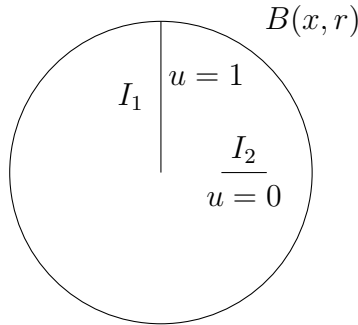


FIGURE 1.

other hand,  $\int_{B(x,r)} \Psi(|\nabla u|) = o(r)$ , for  $H^1$ -a.e.  $x = (x_1, 0)$ . This leads us to the following definition.

**Definition 3.1.** We say that  $E \subset (0, 1)$  is  $(p, \lambda)$ -porous,  $1 \leq p < 2$  and  $\lambda \in \mathbb{R}$ , if for  $H^1$ -a.e.  $x = (x_1, 0) \in E$  there is a sequence of numbers  $r_i > 0$  and a constant  $C_x > 0$  such that  $r_i \rightarrow 0$  as  $i \rightarrow \infty$ , and each interval  $(x_1 - r_i, x_1 + r_i)$  contains an interval  $I_i \subset [0, 1] \setminus E$  with  $H^1(I_i)^{2-p} \log^\lambda(1/H^1(I_i)) \geq C_x r_i$ . We say that  $E$  is  $(2, \lambda)$ -porous if we have the same as above with  $\log^{\lambda-1}(1/H^1(I_i)) \geq C_x r_i$  when  $\lambda < 1$  and  $[\log \log(1/H^1(I_i))]^{-1} \geq C_x r_i$  when  $\lambda = 1$ .

When  $\lambda = 0$ , the above porosity condition is same as that of [Kos99]. Notice that for  $p = 1$ , only the case  $\lambda > 0$  is non-trivial above in the sense that there are no  $(1, \lambda)$ -porous sets when  $\lambda < 0$  and a  $(1, 0)$ -porous set necessarily has length zero.

We begin by showing that porous sets are removable, a part of our main theorem.

**Theorem 3.2.** *If  $E$  is  $(p, \lambda)$ -porous,  $1 \leq p < 2$  and  $\lambda \in \mathbb{R}$ , then  $E$  is  $(p, \lambda)$ -removable. This is also true for  $p = 2$  and  $\lambda \leq 1$ .*

*Proof.* As discussed in our introduction, it suffices to consider functions  $u \in W^{1,\Psi}(B(0, 2) \setminus E) \cap C^1(B(0, 2) \setminus E)$ . First note that for all  $t \geq 0$

$$(3.1) \quad \Psi^{-1}(t) \approx t^{\frac{1}{p}} / \log^{\frac{\lambda}{p}}(e + t),$$

where  $\Psi^{-1}$  is the generalised inverse of  $\Psi$ . Also we have, by the usual covering theorems [Zie89, p.118], that

$$(3.2) \quad \lim_{r \rightarrow 0} \frac{1}{r} \int_{B(x,r)} \Psi(|\nabla u|) dx = 0$$

for  $H^1$ -a.e.  $x \in B(0, 2)$ .

**Case I.**  $1 < p < 2$ . Fix  $x \in E$  so that the upper and lower limits  $u^+(x)$  and  $u^-(x)$  exist and (3.2) holds and also the porosity condition holds for  $x$ . It is enough to prove that  $u^+(x) = u^-(x)$ . Let us assume that  $u^+(x) \neq u^-(x)$ . So, by subtracting a constant, scaling and truncating  $u$ , without any loss of generality we may assume that  $u = 1$  in  $A^+ = \{(x_1, t) : 0 < t < \epsilon\}$  and  $u = 0$  in  $A^- = \{(x_1, t) : -\epsilon < t < 0\}$ . Fix  $r_i < \epsilon$ ,  $I_i$  as in the definition of porosity and write  $I'_i = \{y \in I_i : u(y) \leq \frac{1}{36}\}$  and  $I''_i = I_i \setminus I'_i$ . By symmetry, we may assume that  $H^1(I'_i) \geq \frac{1}{2}H^1(I_i)$ . Fix a ball  $B_0$  of radius  $s_0 = \frac{1}{2}H^1(I_i)$

centred on  $I_i$  with  $B_0 \cap \mathbb{R} \subset I_i$  and another ball  $B'$  of radius  $\frac{1}{2}r_i$  centred on  $A^+$  with  $B' \subset B(x, r_i)^+$ . Here  $B(x, r_i)^+$  denotes the upper half of the ball  $B(x, r_i)$ .

If we have  $\int_{B_0} u \geq 60/81$ , then we consider a cube  $Q_0$  whose sides are parallel to the axes and of side length  $2s_0$  and which contains the ball  $B_0$ . If we assume that  $\int_{Q_0} u \leq 2/3$ , then by the Poincaré inequality we obtain  $\int_{Q_0} |\nabla u(x)| dx \geq cs_0$  for some constant  $c$ , which implies that  $\int_{B(x, r_i)} |\nabla u(x)| dx \geq cr_i$ . Using Jensen's inequality, one obtains

$$\int_{B(x, r_i)} \Psi(|\nabla u(x)|) dx \geq \Psi \left( \int_{B(x, r_i)} |\nabla u(x)| dx \right) \geq \Psi \left( \frac{c}{r_i} \right)$$

which contradicts with (3.2) and concludes the theorem for this particular case. Now, we may assume that  $\int_{Q_0} u \geq 2/3$ . Then by using the Fubini theorem and the fundamental theorem of calculus, we get  $\int_{Q_0} |\nabla u(x)| dx \geq s_0/18^2$  and again using Jensen's inequality we get a contradiction with (3.2). Therefore we assume that  $\int_{B_0} u \leq 60/81$ .

If we have  $\int_{B'} u \leq 61/81$ , then again we consider a cube  $Q'$  whose sides are parallel to the axes and of side length  $r_i$  and which contains  $B'$ . If we assume that  $\int_{Q'} u \geq 64/81$ , then using the Poincaré inequality and Jensen's inequality we get a contradiction with (3.2) as above. Otherwise we use the Fubini theorem and the fundamental theorem of calculus and also Jensen's inequality at the end to conclude the theorem. So now we assume that  $\int_{B'} u \geq 61/81$ .

We use the telescopic argument for the balls  $B'$  and  $B_0$ . This means that we consider a finite number of balls  $B_0, B_1, \dots, B_k = B'$  whose centres lie on the line joining the centres of  $B'$  and  $B_0$  with  $|B_j \cap B_{j+1}| \geq \frac{1}{10}|B_j|$  and the radii increase geometrically so that they form a portion of a cone. We may assume that no point in  $\mathbb{R}^2$  is contained in more than two of these balls. From the construction together with the Poincaré inequality and Hölder's inequality we have

$$\begin{aligned} (3.3) \quad \frac{1}{81} &\leq |u_{B_0} - u_{B'}| \leq \sum_{j=0}^{k-1} |u_{B_j} - u_{B_{j+1}}| \leq \sum_{j=0}^{k-1} cs_j \int_{B_j} |\nabla u(x)| dx \\ &\leq \sum_{j=0}^{k-1} cs_j \left( \int_{B_j} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \sum_{j=0}^{k-1} cs_j^{1-\frac{2}{p}} \left( \int_{B_j} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

where  $s_j$  is the radius of the ball  $B_j$  for  $j = 0, 1, \dots, k-1$ .

First we consider the sub-case  $\lambda \geq 0$ . For this case we split the balls  $B_j$  into “good” part  $B_j^g$  and “bad” part  $B_j^b$  where  $B_j^g = \{x : |\nabla u(x)| \leq \text{diam}(B_j)^{-1/2}\}$  and  $B_j^b = \{x : |\nabla u(x)| > \text{diam}(B_j)^{-1/2}\}$  for  $j = 0, 1, \dots, k-1$ . Using this splitting one obtains

$$(3.4) \quad \frac{1}{81} \leq \sum_{j=0}^{k-1} c s_j^{1/2} + \sum_{j=0}^{k-1} c s_j^{1-\frac{2}{p}} \log^{-\frac{\lambda}{p}} \left( e + s_j^{-1/2} \right) \left( \int_{B_j^b} |\nabla u(x)|^p \log^\lambda (e + |\nabla u(x)|) dx \right)^{\frac{1}{p}} \\ \leq c r_i^{1/2} + c \sum_{j=0}^{k-1} \frac{1}{s_j^{\frac{2-p}{p}} \log^{\frac{\lambda}{p}} (e + s_j^{-1/2})} \left( \int_{B_j^b} \Psi(|\nabla u(x)|) dx \right)^{1/p}.$$

We again use the Hölder’s inequality to obtain

$$(3.5) \quad \frac{1}{81} - c r_i^{\frac{1}{2}} \leq c \left( \sum_{j=0}^{k-1} \frac{1}{s_j^{\frac{2-p}{p-1}} \log^{\frac{\lambda}{p-1}} (e + s_j^{-1/2})} \right)^{1-\frac{1}{p}} \left( \sum_{j=0}^{k-1} \int_{B_j^b} \Psi(|\nabla u(x)|) dx \right)^{\frac{1}{p}}.$$

Since the radii of the balls  $B_j$  are in geometric series, one obtains

$$(3.6) \quad \int_{B(x, r_i)} \Psi(|\nabla u(x)|) dx \geq c s_0^{2-p} \log^\lambda \left( \frac{1}{s_0} \right) \left( \frac{1}{6} - c r_i^{1/2} \right)^p.$$

For the sub-case  $\lambda < 0$ , we apply Jensen’s inequality to the first line of (3.3) and use (3.1) to get

$$\frac{1}{81} \leq \sum_{j=0}^{k-1} c s_j \Psi^{-1} \left( \int_{B_j} \Psi(|\nabla u(x)|) dx \right) \\ \leq \sum_{j=0}^{k-1} \frac{c s_j \left( \int_{B_j} \Psi(|\nabla u(x)|) dx \right)^{1/p}}{\log^{\frac{\lambda}{p}} \left( e + \int_{B_j} \Psi(|\nabla u(x)|) dx \right)}.$$

Let us consider the bigger ball  $B = B(0, 10)$  containing all the balls  $B_j$ ,  $j = 0, 1, \dots, k-1$ .

Now  $\int_{B_j} \Psi(|\nabla u|) dx \leq \int_{B \setminus E} \Psi(|\nabla u|) dx \leq M$  for  $j = 0, 1, \dots, k$ , where  $M$  is a constant independent of  $x$  and  $r_i$ . Apply this estimate and the Hölder’s inequality to the above inequality to obtain

$$(3.7) \quad \frac{1}{81} \leq c \left( \sum_{j=0}^{k-1} \frac{1}{s_j^{\frac{2-p}{p-1}} \log^{\frac{\lambda}{p-1}} (e + M s_j^{-2})} \right)^{1-\frac{1}{p}} \left( \sum_{j=0}^{k-1} \int_{B_j} \Psi(|\nabla u(x)|) dx \right)^{1/p}$$

Consequently,

$$(3.8) \quad \int_{B(x, r_i)} \Psi(|\nabla u(x)|) dx \geq c s_0^{2-p} \log^\lambda \left( \frac{1}{s_0} \right).$$



Taking (3.6) into account we conclude that (3.8) holds both for  $\lambda \geq 0$  and for  $\lambda < 0$ . Recalling that  $s_0 = \frac{1}{2}H^1(I_i)$  and using the porosity condition we get a contradiction with (3.2).

**Case II.**  $p = 1, \lambda \in \mathbb{R}$ . If  $\lambda \leq 0$ , then  $E$  necessarily has vanishing length and removability is clear. For  $\lambda > 0$ , we proceed similarly like in the previous case to obtain from (3.4)

$$\frac{1}{81} - cr_i^{1/2} \leq \sum_{j=0}^{k-1} \frac{c \left( \int_{B_j^b} \Psi(|\nabla u(x)|) dx \right)}{s_j \log^\lambda \left( e + s_j^{-\frac{1}{2}} \right)}.$$

Hence one gets the desired estimate as

$$\int_{B(x, r_i)} \Psi(|\nabla u(x)|) \geq cs_0 \log^\lambda \left( \frac{1}{s_0} \right)$$

and obtains the desired conclusion similarly as in Case I.

**Case III.**  $p = 2, \lambda \leq 1$ . For  $0 < \lambda \leq 1$ , from the inequality (3.5), we have the estimate

$$\frac{1}{81} - cr_i^{1/2} \leq c \left( \sum_{j=0}^{k-1} \frac{1}{\log^\lambda(e + s_j^{-1/2})} \right)^{\frac{1}{2}} \left( \sum_{j=0}^{k-1} \int_{B_j^b} \Psi(|\nabla u(x)|) dx \right)^{\frac{1}{2}},$$

and for  $\lambda < 0$ , from the inequality (3.7), we have the estimate

$$\frac{1}{81} \leq c \left( \sum_{j=0}^{k-1} \frac{1}{\log^\lambda(e + Ms_j^{-2})} \right)^{1/2} \left( \sum_{j=0}^{k-1} \int_{B_j} \Psi(|\nabla u(x)|) dx \right)^{1/2}.$$

Hence we have

$$\int_{B(x, r_i)} \Psi(|\nabla u(x)|) dx \geq c \log^{\lambda-1} \left( \frac{1}{s_0} \right)$$

for  $\lambda < 1$  and

$$\int_{B(x, r_i)} \Psi(|\nabla u(x)|) dx \geq \frac{c}{\log \log \left( \frac{1}{s_0} \right)}$$

for  $\lambda = 1$  and conclude similarly as in Case I to finish the proof.  $\square$

The next theorem shows that  $E$  cannot be removable if the complementary intervals are small. This result will help us to prove the sharpness in Theorem A. For an interval  $I = (a, b)$  and a positive real number  $c$ , we write  $cI$  to denote the interval  $(\frac{a+b}{2} - \frac{c(b-a)}{2}, \frac{a+b}{2} + \frac{c(b-a)}{2})$ . For a rectangle  $W$ , we define  $cW$  in a similar way.

**Theorem 3.3.** *Let  $E \subset (0, 1)$  be compact with  $(0, 1) \setminus E = \bigcup_{j=1}^{\infty} I_j$ , where  $I_j$  are pairwise disjoint open intervals. Suppose that*

(i)  $H^1(E) > 0$  and  $\sum_{j=1}^{\infty} H^1(I_j)^{2-p} \log^{\lambda}(1/H^1(I_j)) < \infty$ , when  $1 < p < 2$ ,  $\lambda \in \mathbb{R}$  or  $p = 1$ ,  $\lambda \geq 0$ ;

(ii)  $H^1((0, 1) \setminus \bigcup_{i=1}^{\infty} H^1(I_i)^{-1/2} I_i) > 0$  and  $\sum_{j=1}^{\infty} \log^{\lambda-1}(1/H^1(I_j)) < \infty$ , when  $p = 2$ ,  $\lambda < 1$ ;

(iii)  $H^1\left((0, 1) \setminus \bigcup_{i=1}^{\infty} \frac{R_j}{H^1(I_j)} I_j\right) > 0$  and  $\sum_{j=1}^{\infty} (\log \log(1/H^1(I_j)))^{-1} < \infty$ , when  $p = 2$ ,  $\lambda = 1$ . Here  $R_j = \exp(-\log(1/H^1(I_j))^{1/2})$ .

Then  $E$  is not  $(p, \lambda)$ -removable.

Notice that (i), for  $p = 1, \lambda = 0$ , shows that there are no  $(1, 0)$ -removable compact sets  $E \subset (0, 1)$  of positive length.

The idea to prove this theorem is to construct a function  $u \in W^{1,\Psi}(\Omega \setminus E)$  for which the two sided limits do not coincide in a subset of  $E$  of positive  $H^1$ -measure.

*Proof.* (i) Let  $\Omega = B(\frac{1}{2}, \frac{1}{2})$ . We define a function  $u$  in  $\Omega \setminus E$  as follows:

$$u(x) = \begin{cases} \min\{\frac{x_2}{d(x,E)}, \frac{1}{\sqrt{2}}\} & \text{if } x_2 \geq 0, \\ 0 & \text{if } x_2 < 0, \end{cases}$$

where  $x_2$  is the second coordinate of  $x$ . Then  $u$  is locally Lipschitz and  $|\nabla u| \leq M < \infty$  almost everywhere in  $\Omega \setminus \bigcup_{j=1}^{\infty} \Delta_j$ , where  $\Delta_j$  is an isosceles right angle triangle in the upper half plane with hypotenuse  $I_j$ . We also have that  $|\nabla u(x)|$  is comparable with  $1/d(x, E)$  when  $x \in \Delta_j$ . Hence, using the fact that  $E$  lies outside  $\Delta_j$  for all  $j$  and also using polar coordinates, we have

$$\int_{\Delta_j} \Psi(|\nabla u|) dx \leq C H^1(I_j)^{2-p} \log^{\lambda} \left( \frac{1}{H^1(I_j)} \right).$$

Then, by the assumption of the theorem, we conclude that  $u \in W^{1,\Psi}(B(0, 2) \setminus E)$ . But when  $x = (x_1, 0) \in E$  we see that  $u^+(x) = 1/\sqrt{2}$  whereas  $u^-(x) = 0$ . It is easy to check that  $u$  cannot be extended to a function in  $W^{1,\Psi}(\Omega)$ .

(ii) Set  $\Omega = (0, 1) \times (-1, 1)$ . Then for every  $I_j$  from our collection, we define

$$W_j = (H^1(I_j)^{-\frac{1}{2}} I_j) \times (-H^1(I_j)^{\frac{1}{2}}, H^1(I_j)^{\frac{1}{2}}).$$

Given  $j$ , we define, for  $x \in \Omega \setminus E$ ,

$$f_j(x) = (|x - x_j| \log(1/H^1(I_j)))^{-1} \chi_{W_j \setminus H^1(I_j)^{\frac{1}{2}} W_j}(x)$$

and  $g(x) = \max_j f_j(x)$ , where  $x_j$  is the centre point of  $I_j$ . Then  $g$  is locally bounded in  $\Omega \setminus E$ . Set  $y = (\frac{1}{2}, \dots, \frac{1}{2}, -1)$  and for every  $x \in \Omega \setminus E$  define

$$u(x) = \inf_{\gamma_x} \int_{\gamma_x} g(x) dH^1,$$

where the infimum is taken over all rectifiable curves joining  $x$  and  $y$  in  $((0, 1) \times [-1, 1]) \setminus E$ .

Then  $u$  is locally Lipschitz in  $\Omega \setminus E$  and we get

$$\begin{aligned} \int_{\Omega \setminus E} \Psi(|\nabla u|) &\leq C \sum_{j=1}^{\infty} \log^{-2} \left( \frac{1}{H^1(I_j)} \right) \int_{W_j \setminus H^1(I_j)^{\frac{1}{2}} W_j} \frac{dx}{|x - x_j|^2} \log^{\lambda} \left( e + \frac{1}{|x - x_j|} \right) \\ &\leq C \sum_{j=1}^{\infty} \log^{\lambda-1} \left( \frac{1}{H^1(I_j)} \right) < \infty, \end{aligned}$$

and consequently  $u \in W^{1,\Psi}(\Omega \setminus E)$ . But  $u \geq 1/2$  in the upper half of  $\Omega$  from the construction whereas  $\lim_{t \rightarrow 0-} u(x', t) = 0$  for all  $x' \in (0, 1) \setminus \bigcup_{i=1}^{\infty} H^1(I_j)^{-1/2} I_j$ , which has positive measure by the assumption. Hence  $E$  is not removable for  $u$ .

(iii) This case is very similar to the previous case. Here we take the functions

$$f_j(x) = \left( \log \log \left( \frac{1}{H^1(I_j)} \right) |x - x_j| \log \left( \frac{1}{|x - x_j|} \right) \right)^{-1} \chi_{W_j \setminus \frac{H^1(I_j)}{R_j} W_j}(x),$$

where

$$W_j = \left( \frac{R_j}{H^1(I_j)} I_j \right) \times (-R_j, R_j).$$

Then we get

$$\begin{aligned} \int_{\Omega \setminus E} \Psi(|\nabla u|) &\leq C \sum_{j=1}^{\infty} \left( \log \log \left( \frac{1}{H^1(I_j)} \right) \right)^{-2} \int_{W_j \setminus \frac{H^1(I_j)}{R_j} W_j} \frac{dx}{|x - x_j|^2 \log \left( \frac{1}{|x - x_j|} \right)} \\ &\leq C \sum_{j=1}^{\infty} \left( \log \log \left( \frac{1}{H^1(I_j)} \right) \right)^{-2} \left( \log \log \left( \frac{1}{H^1(I_j)} \right) - \log \log \left( \frac{1}{R_j} \right) \right) \\ &= C \sum_{j=1}^{\infty} \left( \log \log \left( \frac{1}{H^1(I_j)} \right) \right)^{-1} < \infty. \end{aligned}$$

□

**Proof of Theorem A for  $n=2$ .** Let  $1 < p < 2, \lambda \in \mathbb{R}$  or  $p = 1, \lambda > 0$ . By Theorem 3.2 and Theorem 3.3 it suffices to construct a  $(p, \lambda)$ -porous Cantor set  $E \subset [0, 1]$  of positive length and with  $\sum_{j=1}^{\infty} H^1(I_j)^{2-p} \log^{\lambda-\epsilon}(e + 1/H^1(I_j)) < \infty$  for every  $\epsilon > 0$ , where  $I_j$  are the complementary intervals of  $E$  on  $[0, 1]$ .

We modify the example constructed by Koskela in [Kos99]. The set  $E$  is obtained by the following Cantor construction. Let  $0 < s < \frac{1}{3}$  be a small constant to be determined momentarily. We begin by deleting an open interval of length  $s2^{-\frac{2}{2-p}}$  from the middle of  $[0, 1]$ . We are then left with two closed intervals. We continue the process as follows: if we are left with  $2^{i-1}$  closed intervals, we remove from the middle of each of those intervals an open interval of length  $s2^{-\frac{2i}{2-p}}/i^{\frac{\lambda}{2-p}}$ , provided  $i \in M = \mathbb{N} \setminus \{2^j : j \in \mathbb{N}\}$ , and if we are left with  $2^{2j-1}$  closed intervals, we remove an open interval of length  $s2^{-\frac{2j}{2-p}}/2^{\frac{j\lambda}{2-p}}$ . By induction we obtain a nested sequence of closed intervals. We define  $E$  as the intersection of all these closed intervals. The total length of the removed intervals is

$$\sum_{i \in M} 2^{i-1} \frac{s2^{-\frac{2i}{2-p}}}{i^{\frac{\lambda}{2-p}}} + \sum_{j \in \mathbb{N}} 2^{2j-1} \frac{s2^{-\frac{2j}{2-p}}}{2^{\frac{j\lambda}{2-p}}} < \infty.$$

This sum can be made strictly less than 1 by choosing  $s$  sufficiently small and so  $E$  has positive length. We have constructed the set  $E$  in such a way that for any  $x \in E$  and  $j \geq 1$ , we get a complementary interval  $J_j$  of length  $s2^{-\frac{2j}{2-p}}/2^{\frac{j\lambda}{2-p}}$  and with  $d(x, J_j) \leq 2^{-2j}$ . Hence  $(p, \lambda)$ -porosity of  $E$  follows. Finally, to see that  $E$  is not  $(p, \lambda - \epsilon)$ -removable, we have to check the convergence of the sum  $\sum_{j=1}^{\infty} H^1(I_j)^{2-p} \log^{\lambda-\epsilon}(1/H^1(I_j))$  for  $\epsilon > 0$ , which in this case turns out to be

$$\begin{aligned} \sum_{j=1}^{\infty} H^1(I_j)^{2-p} \log^{\lambda-\epsilon}(1/H^1(I_j)) &= \sum_{i \in M} 2^i \left( \frac{s2^{-\frac{2i}{2-p}}}{i^{\frac{\lambda}{2-p}}} \right)^{2-p} \log^{\lambda-\epsilon} \left( e + \frac{2^{\frac{2i}{2-p}}}{si^{-\frac{\lambda}{2-p}}} \right) \\ &\quad + \sum_{j \in \mathbb{N}} 2^{2j} \left( \frac{s2^{-\frac{2j}{2-p}}}{2^{\frac{j\lambda}{2-p}}} \right)^{2-p} \log^{\lambda-\epsilon} \left( e + \frac{2^{\frac{2j}{2-p}}}{s2^{-\frac{j\lambda}{2-p}}} \right) \\ &\leq C \sum_{i \in M} \frac{2^{-i}}{i^{\epsilon}} + \sum_{j \in \mathbb{N}} \frac{1}{2^{j\epsilon}} \end{aligned} \tag{3.9}$$

and hence the sum is finite for every  $\epsilon > 0$  (note that the sum does not converge when  $\epsilon$  is zero).

Let  $p = 2, \lambda < 1$ . We remove open intervals of length  $s2^{-i} \exp(-2^{\frac{i}{1-\lambda}})$  when we are left with  $2^{i-1}$  closed intervals for  $i \in \mathbb{N}$  and then it is easy to verify the porosity condition

and also the convergence of the series  $\sum_{j=1}^{\infty} \log^{\lambda-1-\epsilon}(1/H^1(I_j))$  for every  $\epsilon > 0$ .

Now let  $p = 2$ ,  $\lambda = 1$ . Here we remove open intervals of length  $s2^{-i} \exp(-\exp(2^i))$  when we are left with  $2^{i-1}$  closed intervals for  $i \in \mathbb{N}$ . Then one has to check that  $\sum_{j=1}^{\infty} (\log \log(1/H^1(I_j)))^{-1} = \infty$  but  $\sum_{j=1}^{\infty} \log^{-\epsilon}(1/H^1(I_j)) < \infty$  for every  $\epsilon > 0$ , which is easy to do. This completes the proof of the main theorem in the plane case.  $\square$

#### 4. THE HIGHER DIMENSIONAL CASE

Similarly to the case  $n = 2$ , we would like to consider the one sided limits and to show that they coincide. But since line segments have  $p$ -capacity zero for  $p \leq n - 1$ , one can not use the same argument as in the plane case. In [Kos99], the author has used  $p$ -harmonic functions to overcome this problem. We do not know how to use  $\Psi$ -harmonic functions in our setting. Instead of this we extend the restriction of our function to the upper (or lower) half space by reflection to the entire space and take a quasicontinuous representative of this  $W_{\text{loc}}^{1,1}$ -Sobolev function to reduce the problem to the following. If a function  $u \in W^{1,\Psi}$  is such that  $u_{B'} \geq 61/81$  and  $u \leq 1/36$  on half of  $A$  (see Figure 2), then using a chaining argument and Poincaré inequality we get a lower bound

$$\int_{B(x,r)} \Psi(|\nabla u|) \geq cs^{n-p} \log^{\lambda} \left( \frac{1}{s} \right),$$

for  $1 \leq p < n - 1$ , where  $s = \text{diam}(A)$  and similar estimates hold for different pairs of  $(p, \lambda)$ . But on the other hand, we know that  $\int_{B(x,r)} \Psi(|\nabla u|) = o(r^{n-1})$  for  $H^{n-1}$ -a.e.

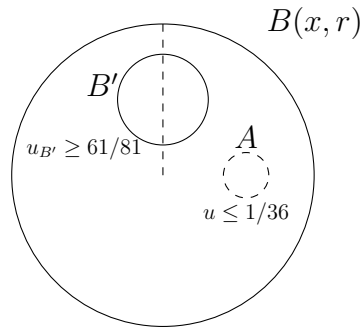


FIGURE 2.

$x$ . Then again the definition of the porosity comes in a natural way. Before defining the porosity condition, we prove a lemma which allows us to consider even a continuum rather than a ball in the definition of porosity for some cases.

**Lemma 4.1.** *Let  $0 < r < 1$ . Denote by  $B(0, r)^+$  the upper half of the  $n$ -dimensional ball  $B(0, r)$  of radius  $r$ . Let  $F \subset B(0, r) \cap \mathbb{R}^{n-1}$  be compact with  $H^1(F) \geq r/3$ . Let  $\mathcal{W}$  be a Whitney decomposition of  $B(0, r)^+$ . Suppose  $u \in C^1(B(0, r)^+) \cap C(B(0, r)^+ \cup F)$  satisfies  $u = 0$  on  $F$  and  $\int_{Q_1} u \geq \frac{1}{2}$ , where  $Q_1 \in \mathcal{W}$  is a largest cube contained in  $B(0, r)^+$ . Then*

$$\int_{B(0, r)^+} |\nabla u|^p \log^\lambda(e + |\nabla u|) \geq \begin{cases} Cr^{n-p} \log^\lambda\left(\frac{1}{r}\right) & \text{when } n-1 < p \leq n, \lambda \in \mathbb{R}, \\ Cr \log^{\lambda-(n-2)}\left(\frac{1}{r}\right) & \text{when } p = n-1, \lambda > n-2. \end{cases}$$

*Proof.* First note that a change of variables  $y = x/r$  gives the estimate

$$\int_{B(0, r)^+} |\nabla u(x)|^p \log^\lambda(e + |\nabla u(x)|) dx = r^{n-p} \int_{B(0, 1)^+} |\nabla v(y)|^p \log^\lambda\left(e + \frac{|\nabla v(y)|}{r}\right) dy,$$

where  $v(y) = u(ry)$  satisfies  $v \in C^1(B(0, 1)^+) \cap C(B(0, 1)^+ \cup F')$ ,  $v = 0$  on  $F'$ . Here  $F' \subset B(0, 1) \cap \mathbb{R}^{n-1}$  is the transformed compact set with  $H^1(F') \geq 1/3$ . Denote  $\mathcal{W}'$  the collection of cubes from  $\mathcal{W}$  after rescaling. The function  $v$  also satisfies  $\int_{Q'} v \geq \frac{1}{2}$ , where  $Q' \subset B(0, 1)^+$  is the corresponding transformed cube from the collection  $\mathcal{W}_1$ .

Since  $H^1(F') > 0$ , Frostman's lemma (p.112 of [Mat95]) implies that there exists a Radon measure  $\mu$  supported in  $F'$  so that  $\mu(B(x, r)) \leq r$  for all  $x \in \mathbb{R}^n$  and all  $r > 0$  and that  $\mu(F') \geq cH^1(F') \geq c/3$ , where  $c$  is a positive constant depending only on  $n$ .

For  $x \in F'$ , denote by  $I_x$  the line segment joining  $x$  to the centre of  $Q'$  and let  $\mathcal{Q}(x)$  consist of all the cubes  $Q \in \mathcal{W}'$  such that  $I_x$  intersects the cube  $Q$ . Now we use the Poincaré inequality for the chain of cubes to obtain

$$\frac{1}{2} \leq |v(x) - v_{Q_1}| \leq C \sum_{Q \in \mathcal{Q}(x)} \ell(Q) \left( \int_Q |\nabla v|^p \right)^{\frac{1}{p}},$$

where  $\ell(Q)$  denotes the edge length of  $Q$ . We split the cubes  $Q \in \mathcal{Q}(x)$  into “good” part  $Q^g$  and “bad” part  $Q^b$  where  $Q^g = \{x : |\nabla v(x)| \leq \ell(Q)^{-1/2}\}$  and  $Q^b = \{x : |\nabla v(x)| > \ell(Q)^{-1/2}\}$ . Using this splitting similarly to the inequality (3.4), we rewrite the above inequality as

$$1 \leq C \sum_{Q \in \mathcal{Q}(x)} \frac{\ell(Q)^{1-\frac{n}{p}}}{\log^{\frac{\lambda}{p}}\left(e + \frac{\ell(Q)^{-\frac{1}{2}}}{r}\right)} \left( \int_Q |\nabla v|^p \log^\lambda\left(e + \frac{|\nabla v|}{r}\right) \right)^{\frac{1}{p}}$$

for  $\lambda > 0$ . (For  $\lambda < 0$  we use Jensen's inequality similarly to the proof of Theorem 3.2 to get the above inequality.) By integrating with respect to  $\mu$  and using the Fubini theorem

and Hölder's inequality we get

$$\begin{aligned}
\mu(F') &\leq C \int_{F'} \sum_{Q \in \mathcal{Q}(x)} \frac{\ell(Q)^{1-\frac{n}{p}}}{\log^{\frac{\lambda}{p}} \left( e + \frac{\ell(Q)^{-\frac{1}{2}}}{r} \right)} \left( \int_Q |\nabla v|^p \log^\lambda \left( e + \frac{|\nabla v|}{r} \right) \right)^{\frac{1}{p}} d\mu(x) \\
&\leq C \sum_{Q \in \mathcal{W}'} \frac{\ell(Q)^{1-\frac{n}{p}}}{\log^{\frac{\lambda}{p}} \left( e + \frac{\ell(Q)^{-\frac{1}{2}}}{r} \right)} \left( \int_Q |\nabla v|^p \log^\lambda \left( e + \frac{|\nabla v|}{r} \right) \right)^{\frac{1}{p}} \mu(S(Q)) \\
&\leq C \left( \sum_{Q \in \mathcal{W}'} \int_Q |\nabla v|^p \log^\lambda \left( e + \frac{|\nabla v|}{r} \right) \right)^{\frac{1}{p}} \left( \sum_{Q \in \mathcal{W}'} \frac{\ell(Q)^{\frac{p-n}{p-1}} \mu(S(Q))^{\frac{p}{p-1}}}{\log^{\frac{\lambda}{p-1}} \left( e + \frac{\ell(Q)^{-\frac{1}{2}}}{r} \right)} \right)^{1-\frac{1}{p}},
\end{aligned}$$

where  $S(Q) \subset F'$  denotes the “shadow” of a cube  $Q$ , i.e. those points  $x \in F'$  for which  $I_x \cap Q \neq \emptyset$ . Furthermore, denote by  $\mathcal{W}_j$  all the cubes in the  $j$ th generation of Whitney cubes, i.e.  $\mathcal{W}_j$  consists of the cubes  $Q \in \mathcal{W}'$  of edge length between  $2^{-j}$  and  $2^{-(j+1)}$ . We deduce that

$$\begin{aligned}
\mu(F')^p &\leq C \int_{B(0,1)^+} |\nabla v|^p \log^\lambda \left( e + \frac{|\nabla v|}{r} \right) \left( \sum_{j=1}^{\infty} \sum_{Q \in \mathcal{W}_j} \frac{2^{-j\frac{p-n}{p-1}} \mu(S(Q))^{\frac{p}{p-1}}}{\log^{\frac{\lambda}{p-1}} \left( e + \frac{2^{\frac{j}{2}}}{r} \right)} \right)^{p-1} \\
&\leq C \mathcal{I} \left( \sum_{j=1}^{\infty} \frac{2^{-j\frac{p-n}{p-1}} \max_{Q \in \mathcal{W}_j} \mu(S(Q))^{\frac{1}{p-1}}}{\log^{\frac{\lambda}{p-1}} \left( e + \frac{2^{\frac{j}{2}}}{r} \right)} \sum_{Q \in \mathcal{W}_j} \mu(S(Q)) \right)^{p-1} \\
&\leq C \mu(F')^{p-1} \mathcal{I} \left( \sum_{j=1}^{\infty} \frac{2^{-j\frac{p+1-n}{p-1}}}{\log^{\frac{\lambda}{p-1}} \left( e + \frac{2^{\frac{j}{2}}}{r} \right)} \right)^{p-1},
\end{aligned}$$

where we have denoted the integral  $\int_{B(0,1)^+} |\nabla v|^p \log^\lambda \left( e + \frac{|\nabla v|}{r} \right)$  by  $\mathcal{I}$ . Using the fact that  $\mu(F') \geq c/3$  and estimating the sum in the right hand side of the above inequality, we have

$$1 \leq C \mathcal{I} \left( \log^{-\frac{\lambda}{p-1}} \left( \frac{1}{r} \right) \right)^{p-1},$$

when  $n-1 < p \leq n, \lambda \in \mathbb{R}$ ;

$$1 \leq C \mathcal{I} \left( \log^{-\frac{\lambda}{n-2}+1} \left( \frac{1}{r} \right) \right)^{n-2},$$

when  $p = n - 1, \lambda > n - 2$ ; which implies that

$$\mathcal{I} \geq C \log^\lambda \left( \frac{1}{r} \right) \text{ or } C \log^{\lambda-(n-2)} \left( \frac{1}{r} \right)$$

according to  $n - 1 < p \leq n, \lambda \in \mathbb{R}$  or  $p = n - 1, \lambda > n - 2$ . This proves the lemma.  $\square$

**Definition 4.2.** We say that  $E \subset \mathbb{R}^{n-1}$  is  $(p, \lambda)$ -porous, if for  $H^{n-1}$ -a.e.  $x \in E$ , there is a sequence of  $r_i > 0$  and a constant  $c_x > 0$  such that  $r_i \rightarrow 0$  as  $i \rightarrow \infty$  and each  $(n - 1)$ -dimensional ball  $B(x, r_i)$  contains

- (i) a ball  $B_i \subset B(x, r_i) \setminus E$  of radius  $R_i$  with  $R_i^{n-p} \log^\lambda \left( \frac{1}{R_i} \right) \geq C_x r_i^{n-1}$  when  $1 \leq p < n - 1$  and  $\lambda \in \mathbb{R}$ ,
- (ii) a ball  $B_i \subset B(x, r_i) \setminus E$  of radius  $R_i$  with  $R_i \log^\lambda \left( \frac{1}{R_i} \right) \geq C_x r_i^{n-1}$  when  $p = n - 1$  and  $\lambda \leq n - 2$ ,
- (iii) a continuum  $F_i \subset B(x, r_i) \setminus E$  of diameter  $R_i$  with  $R_i \log^{\lambda-(n-2)} \left( \frac{1}{R_i} \right) \geq C_x r_i^{n-1}$  when  $p = n - 1$  and  $\lambda > n - 2$ ,
- (iv) a continuum  $F_i \subset B(x, r_i) \setminus E$  of diameter  $R_i$  with  $R_i^{n-p} \log^\lambda \left( \frac{1}{R_i} \right) \geq C_x r_i^{n-1}$  when  $n - 1 < p < n$  and  $\lambda \in \mathbb{R}$ ,
- (v) a continuum  $F_i \subset B(x, r_i) \setminus E$  of diameter  $R_i$  with  $\log^{\lambda-(n-1)} \left( \frac{1}{R_i} \right) \geq C_x r_i^{n-1}$  when  $p = n$  and  $\lambda < n - 1$ ,
- (vi) a continuum  $F_i \subset B(x, r_i) \setminus E$  of diameter  $R_i$  with  $\left( \log \log \left( \frac{1}{R_i} \right) \right)^{1-n} \geq C_x r_i^{n-1}$  when  $p = n$  and  $\lambda = n - 1$ .

Again, the definition of porosity is same as in [Kos99] for  $\lambda = 0$ .

Notice that we have replaced the round holes by holes of suitable diameter in some cases and there is a change in the power of the logarithmic term for different  $p$  and also there is a mismatch in the power of the logarithmic term for different  $\lambda$  when  $p = n - 1$ . Again we will ignore the case  $p = 1, \lambda \leq 0$  as in the planar case because of the same reason.

To prove that porous sets are removable, we need help of the following lemma.

**Theorem 4.3.** *If  $E$  is  $(p, \lambda)$ -porous,  $1 < p < n, \lambda \in \mathbb{R}$  or  $p = 1, \lambda > 0$ , then  $E$  is  $(p, \lambda)$ -removable. This also holds when  $p = n, \lambda \leq n - 1$ .*



*Proof.* Let  $E \subset I^{n-1} = (0, 1)^{n-1}$  and  $u \in W^{1,\Psi}(B(0, 2) \setminus E) \cap C^1(B(0, 2) \setminus E)$ . As in the planar case, it suffices to show that

$$(4.1) \quad \int_{B^n(x, r_i)} \Psi(|\nabla u(x)|) dx \geq C_x r_i^{n-1}$$

for all large enough  $i$  whenever  $x = (x_1, \dots, x_{n-1}, 0) \in E$  is such that the one-sided limits do not coincide at  $x$  and the porosity condition holds at  $x$ . Here  $B^n(x, r_i)$  is the  $n$ -dimensional ball corresponding to the  $(n-1)$ -dimensional ball  $B(x, r_i)$  from the porosity condition. By symmetry and porosity we may assume that the upper limit is 1, the lower limit is 0 and  $u \leq \frac{1}{36}$  in a set  $A \subset B_i$  with  $H_\infty^{n-1}(A) \geq \frac{1}{2}H_\infty^{n-1}(B_i)$  or in a compact set  $A \subset F_i$  with  $H_\infty^1(A) \geq \frac{1}{3}H_\infty^1(F_i)$ .

Unlikely to the proof of Theorem 3.2, to get a ball  $B'$  centred on  $A^+ = \{(x_1, \dots, x_{n-1}, t) : 0 < t < r_i\}$  with  $\int_{B'} u \geq 61/81$ , one needs to do something else as  $p$ -capacity of a line segment in  $\mathbb{R}^n$  is zero for  $p \leq n-1$ . Towards this end, we prove that  $\lim_{i \rightarrow \infty} u_{\hat{B}_i}$  exists for  $\hat{B}_i = B((x_1, \dots, x_{n-1}, r_i/2), r_i/2)$  and is equal to  $\lim_{t \rightarrow 0+} u(x_1, \dots, x_{n-1}, t)$  for  $H^{n-1}$ -a.e  $x \in E$ .

Let  $\epsilon > 0$ . By reflection we obtain a function  $v \in W^{1,1}((0, 1)^n)$  which coincides with  $u$  in the upper half plane. From the 1-quasicontinuity of the precise representative of  $v$  (for details see section 4.8 of [EG92]), we know that  $\lim_{i \rightarrow \infty} v_{\hat{B}_i}$  exists outside a set  $V$  with  $\text{cap}_1(V) \leq \epsilon$ . Actually, [EG92] considers balls centred at  $x$ . However the usual Poincaré inequality gives this stronger statement outside an additional set of vanishing  $H^{n-1}$ -measure. Let

$$F = \{x \in E : \lim_{i \rightarrow \infty} v_{\hat{B}_i} \neq \lim_{t \rightarrow 0+} v(x_1, \dots, x_{n-1}, t)\}.$$

Since Hausdorff measure does not increase under projection, we have that  $H^{n-1}(F) = 0$ . Now, assuming that  $i$  is large enough we can take a ball  $B' \subset B^n(x, r_i)^+$  of radius  $\frac{1}{2}r_i$  centred on  $A^+$  with  $\int_{B'} u \geq 61/81$ .

Fix a ball  $B_0$ , the  $n$ -dimensional ball corresponding to  $B_i$ , when in case (i) or (ii) of Definition 4.2. For the cases (iii)-(vi) in Definition 4.2 we fix a ball  $B_0$  of radius equal to the diameter of  $F_i$  such that  $F_i \subset \overline{B_0} \cap \mathbb{R}^{n-1}$ . Suppose that  $\int_{B_0^+} u \leq 60/81$ . Then we use the telescopic argument for the two balls  $B_0$  and  $B'$  similarly to the proof of Theorem 3.2

to get the lower bound

$$(4.2) \quad \int_{B^n(x, r_i)} \Psi(|\nabla u(x)|) dx \geq \begin{cases} C \log^{\lambda-(n-1)} \left( \frac{1}{R_i} \right) & \text{when } p = n, \lambda < n-1, \\ C \left( \log \log \left( \frac{1}{R_i} \right) \right)^{1-n} & \text{when } p = n, \lambda = n-1, \\ C R_i^{n-p} \log^\lambda \left( \frac{1}{R_i} \right) & \text{otherwise.} \end{cases}$$

Suppose then that  $f_{B_0^+} u \geq 60/81$ . In the case of (i)-(ii) in Definition 4.2 we use the Poincaré inequality, the Fubini theorem, the fundamental theorem of calculus and Jensen's inequality similarly to the proof of Theorem 3.2 to get

$$\int_{B^n(x, r_i)} \Psi(|\nabla u(x)|) dx \geq \Psi \left( \frac{C}{r_i} \right),$$

which contradicts with the fact that  $\lim_{r_i \rightarrow 0} \frac{1}{r_i^{n-1}} \int_{B^n(x, r_i)} \Psi(|\nabla u(x)|) dx = 0$  for  $H^{n-1}$ -a.e.  $x \in B(0, 2)$ . For (iii)-(vi) in Definition 4.2 we apply Lemma 4.1 to  $\frac{81}{120}u$  to conclude that

$$(4.3) \quad \int_{B^n(x, r_i)} \Psi(|\nabla u(x)|) dx \geq \begin{cases} C R_i^{n-p} \log^\lambda \left( \frac{1}{R_i} \right) & \text{when } n-1 < p < n, \lambda \in \mathbb{R} \text{ or} \\ & p = n, \lambda \leq n-1, \\ C R_i \log^{\lambda-(n-2)} \left( \frac{1}{R_i} \right) & \text{when } p = n-1, \lambda > n-2. \end{cases}$$

Taking the respective minimums of the two inequalities (4.2) and (4.3) and using the definition of porosity we get the inequality (4.1). This completes the proof.  $\square$

Next we give sufficient conditions for a set to be non-removable.

**Theorem 4.4.** *Let  $I^{n-1} \setminus E = \bigcup_{i=1}^\infty Q_i$ ,  $I = (0, 1)$ , where  $Q_i$ 's are pairwise disjoint open rectangles of length  $r_i$  of one edge and of length  $\sqrt{2}r_i^2$  of other edges in  $I^{n-1}$  when  $p = n-1, \lambda > n-2$  and  $Q_i$ 's are pairwise disjoint open cubes for the other values of the pair  $(p, \lambda)$ . Suppose that*

- (i)  $H^{n-1}(I^{n-1} \setminus \bigcup_{i=1}^\infty 2Q_i) > 0$  and  $\sum_{i=1}^\infty (\text{diam } Q_i)^{n-p} \log^\lambda(1/\text{diam } Q_i) < \infty$ , when  $1 < p < n-1, \lambda \in \mathbb{R}$  or  $p = 1, \lambda \geq 0$  or  $n-1 < p < n, \lambda \in \mathbb{R}$  or  $p = n-1, \lambda \leq n-2$ ;
- (ii)  $H^{n-1}(I^{n-1} \setminus \bigcup_{i=1}^\infty 2Q_i) > 0$  and  $\sum_{i=1}^\infty \text{diam } Q_i \log^{\lambda-(n-2)}(1/\text{diam } Q_i) < \infty$ , when  $p = n-1, \lambda > n-2$ ;
- (iii)  $H^{n-1}(I^{n-1} \setminus \bigcup_{i=1}^\infty (\text{diam } Q_i)^{-\frac{1}{2}} Q_i) > 0$  and  $\sum_{i=1}^\infty \log^{\lambda-(n-1)}(1/\text{diam } Q_i) < \infty$ , when  $p = n, \lambda < n-1$ ;
- (iv)  $H^{n-1}(I^{n-1} \setminus \bigcup_{i=1}^\infty (\frac{R_i}{\text{diam } Q_i} Q_i) > 0$  and  $\sum_{i=1}^\infty (\log \log(1/\text{diam } Q_i))^{1-n} < \infty$ , when

$p = n, \lambda = n - 1$ . Here  $R_i = \exp(-\log(1/\text{diam } Q_i)^{1/2})$ .

Then  $E$  is not  $(p, \lambda)$ -removable.

*Proof.* (i) Set  $\Omega = I^{n-1} \times (-1, 1)$ . Define  $W_i = (2Q_i) \times (-\text{diam } Q_i, \text{diam } Q_i)$  and  $f_i(x) = (\text{diam } Q_i)^{-1} \chi_{W_i}(x)$  for  $x \in \Omega \setminus E$  and for every  $i$ . For every  $x \in \Omega \setminus E$  define  $g(x) = \max_i f_i(x)$  and  $u(x) = \inf_{\gamma_x} \int_{\gamma_x} g(x) dH^1$ , where the infimum is taken over all the rectifiable curves that join  $x$  to  $y = (\frac{1}{2}, \dots, \frac{1}{2}, -1)$  in  $(I^{n-1} \times [-1, 1]) \setminus E$ . To see that  $u \in W^{1, \Psi}(\Omega \setminus E)$ , observe that

$$\int_{\Omega \setminus E} \Psi(|\nabla u|) dx \leq C \sum_{i=1}^{\infty} (\text{diam } Q_i)^{n-p} \log^\lambda(1/\text{diam } Q_i) < \infty.$$

As  $u \geq 1$  in the upper half of  $\Omega$  and  $\lim_{t \rightarrow 0^-} u(x', t) = 0$  for all  $x' \in I^{n-1} \setminus (\bigcup_{i=1}^{\infty} 2Q_i)$ ,  $E$  is not removable for  $u$ .

(ii) Set  $\Omega = I^{n-1} \times (-1, 1)$ . We rotate  $Q_i$  to form a cylinder of revolution in  $\mathbb{R}^n$  of length  $r_i$  and of radius of base  $r_i^2$  and denote its axis by  $J_i$ . Let  $A_i$  denote the cylindrical annulus of length  $r_i$ , inner radius  $r_i^2$  and outer radius  $r_i$ . Now construct  $W_i$  by closing the two faces of the inner cylinder by half balls of radius  $r_i^2$  and also two faces of the outer cylinder by half balls of radius  $r_i$ , i.e.,  $W_i$  is a kind of two-sided Thermos flask (see Figure 3). For every  $i$ , we define

$$u_i(x) = \begin{cases} \frac{\log\left(\frac{1}{d(x, J_i)}\right) - \log\left(\frac{1}{r_i}\right)}{\log\left(\frac{1}{r_i^2}\right) - \log\left(\frac{1}{r_i}\right)} & \text{if } x \in W_i, \\ 1 & \text{if } x \text{ lies inside the inner rounded cylinder,} \\ 0 & \text{if } x \text{ lies outside the outer rounded cylinder.} \end{cases}$$

Notice that  $u_i$  is Lipschitz for each  $i$ . Define  $u'(x) = \max_j u_j(x)$  and for every  $x = (x_1, \dots, x_n) \in \Omega \setminus E$ ,

$$u(x) = \begin{cases} u'(x) & \text{if } x_n > 0, \\ 1 & \text{if } x_n < 0, \\ 1 & \text{if } x_n = 0 \text{ and } x \in Q_i \text{ for some } i. \end{cases}$$

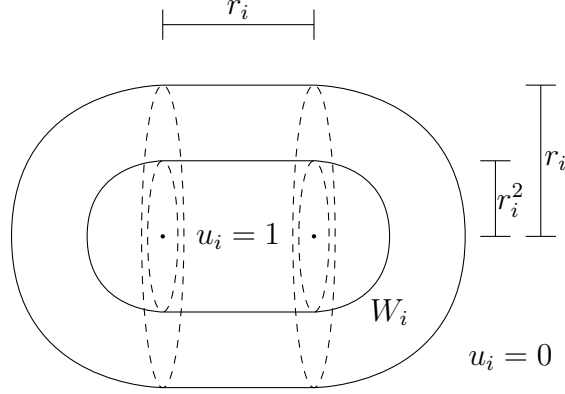


FIGURE 3.

Then by calculating the gradient and using polar coordinates, we obtain

$$\begin{aligned}
\int_{\Omega \setminus E} \Psi(|\nabla u|) dx &= \int_{\Omega} |\nabla u|^{n-1} \log^{\lambda}(e + |\nabla u|) dx \\
&\leq C \sum_{i=1}^{\infty} \frac{r_i}{\left(\log\left(\frac{1}{r_i^2}\right) - \log\left(\frac{1}{r_i}\right)\right)^{n-1}} \int_{r_i^2}^{r_i} \frac{t^{n-2}}{t^{n-1}} \log^{\lambda}\left(\frac{1}{t}\right) dt \\
&\leq C \sum_{i=1}^{\infty} r_i \log^{\lambda-(n-2)}\left(\frac{1}{r_i}\right) < \infty.
\end{aligned}$$

Hence  $u \in W^{1,\Psi}(\Omega \setminus E)$ , but  $u$  can not be extended as a Sobolev function in  $\Omega$ .

Cases (iii) and (iv) can be proved similarly as cases (ii) and (iii) of Theorem 3.3, respectively.  $\square$

**Proof of Theorem A for  $n \geq 3$ .** By Theorem 4.3 and Section 3 it suffices to construct a  $(p, \lambda)$ -porous compact set  $E \subset [0, 1]^{n-1} = I^{n-1}$  of positive  $H^{n-1}$ -measure such that  $E$  is not  $(p, \lambda - \epsilon)$ -removable for any  $\epsilon > 0$ .

Let first  $1 < p < n - 1, \lambda \in \mathbb{R}$  or  $p = 1, \lambda > 0$  or  $n - 1 < p < n, \lambda \in \mathbb{R}$  or  $p = n - 1, \lambda \leq n - 2$ . Set  $A = (n - 1)/(n - p)$ . Let  $l_0 = 1$ . We begin by deleting a cube  $Q_0$  of edge length  $s2^{-2A}$  from the centre of  $I^{n-1}$ , where  $\frac{1}{4} \leq s \leq \frac{1}{2}$  can change its value in every stage. We subdivide  $I^{n-1} \setminus Q_0$  into cubes of different sizes:  $2^{n-1}$  of them of size  $l_1 = \frac{1}{2}(1 - s2^{-2A})$  and the rest of size  $s2^{-2A}$ . The cubes of size  $s2^{-2A}$  correspond to translating the central cube along the coordinate directions. This determines the value of  $s$  at this stage as we need  $2^{2A}/s$  to be an odd integer. See Figure 4. Write  $\mathcal{W}^1$  for the collection of all the cubes in the first subdivision. Then we delete a cube of edge length

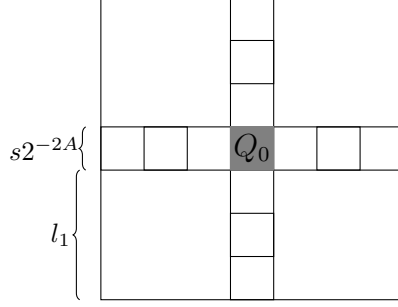


FIGURE 4.

$s2^{-4A}/2^{\frac{2\lambda}{n-p}}$  from the centre of each cube  $Q_{l_1}$  in  $\mathcal{W}^1$  whose size is at least  $\frac{1}{2}l_1$ . Write  $\mathcal{W}^2$  for the cubes in  $\mathcal{W}^1$  whose edge lengths are less than  $\frac{1}{2}l_1$  and the cubes obtained from the subdivision of the cubes subject to the central deletion. The subdivision of such a cube results in cubes of two sizes:  $2^{n-1}$  cubes of size  $l_2 = \frac{1}{2}(\ell(Q_{l_1}) - s2^{-4A}/2^{\frac{2\lambda}{n-p}})$ , the rest of size  $s2^{-4A}/2^{\frac{2\lambda}{n-p}}$ . We repeat the construction in the following way: at stage  $i-1$ , for  $i \in M = \mathbb{N} \setminus \{2^j : j \in \mathbb{N}\}$ , we delete a cube of size  $s2^{-2iA}/i^{\frac{\lambda}{n-p}}$  from the centre of each cube in  $\mathcal{W}^{i-1}$  of size at least  $\frac{1}{2}l_{i-1}$  but when  $i = 2^j$ ,  $j \in \mathbb{N}$ , we delete a cube of size  $s2^{-2^jA}/2^{\frac{j\lambda}{n-p}}$  from the centre of each cube in  $\mathcal{W}^{2^j-1}$  of size at least  $\frac{1}{2}l_{2^j-1}$ . Then the set

$$E = \left( \bigcap_{i \in M} \mathcal{W}^i \right) \cap \left( \bigcap_{j \in \mathbb{N}} \mathcal{W}^{2^j} \right)$$

is clearly  $(p, \lambda)$ -porous and finiteness of the following sum, which follows similarly as in (3.9),

$$\sum_{i \in M} 2^{i(n-1)} \left( \frac{2^{-2iA}}{i^{\frac{\lambda}{n-p}}} \right)^{n-p} \log^{\lambda-\epsilon} \left( 2^{2iA} i^{\frac{\lambda}{n-p}} \right) + \sum_{j \in \mathbb{N}} 2^{2^j(n-1)} \left( \frac{2^{-2^jA}}{2^{\frac{j\lambda}{n-p}}} \right)^{n-p} \log^{\lambda-\epsilon} \left( 2^{2^jA} 2^{\frac{j\lambda}{n-p}} \right)$$

gives the non-removability for every  $\epsilon > 0$ .

Let then  $p = n$  and  $\lambda < n-1$ . We begin by deleting a cube  $Q_1$  of edge length  $s2^{-1} \exp(-2^{\frac{1}{n-1-\lambda}})$  and then let  $l_1 = \frac{1}{2}(1 - s2^{-1} \exp(-2^{\frac{1}{n-1-\lambda}}))$ . In the  $(i-1)$ -th step, for  $i \in \mathbb{N}$ , we delete a cube of edge length  $s2^{-i} \exp(-2^{\frac{i}{n-1-\lambda}})$  from the centre of each cube whose edge length is at least  $\frac{1}{2}l_{i-1}$ . Then we take the set  $E$  as the intersection of the collections of the remaining cubes as before and the rest is easy to verify.

When  $p = n$  and  $\lambda = n-1$ , we delete a cube of edge length  $s2^{-i} \exp(\exp(2^i))$  from the centre of each cube whose edge length is at least  $\frac{1}{2}l_{i-1}$ .

Finally, let  $p = n-1$ ,  $\lambda > n-2$ . Again let  $l_0 = 1$ . Here we begin by deleting a rectangle

$Q_0$  of length  $s2^{-2(n-1)}$  of one edge and of length  $(s2^{-2(n-1)})^2$  of other edges from the centre of  $I^{n-1}$ . We subdivide  $I^{n-1} \setminus Q_0$  into cubes of different sizes:  $2^{n-1}$  of them of size  $l_1 = \frac{1}{2}(1 - (s2^{-2(n-1)})^2)$  and of rest of size  $(s2^{-2(n-1)})^2$ . At this stage,  $s$  can be chosen such that  $2^{2(n-1)}/s$  is an odd integer. See Figure 5. Write  $\mathcal{W}^1$  for the collection of all cubes in the first subdivision. We repeat the construction in the following manner: at stage  $i - 1$ ,

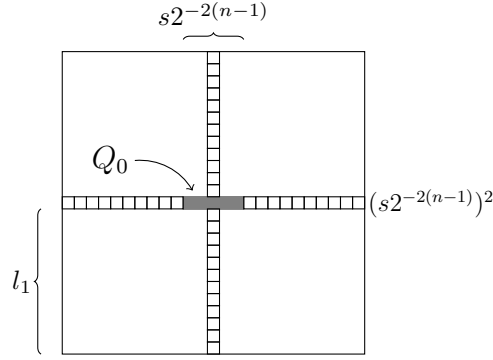


FIGURE 5.

for  $i \in M = \mathbb{N} \setminus \{2^j : j \in \mathbb{N}\}$ , we delete a rectangle of length  $s2^{-2i(n-1)}/i^{\lambda-(n-2)}$  of one edge and of length  $(s2^{-2i(n-1)}/i^{\lambda-(n-2)})^2$  of other edges from the centre of each cube in  $\mathcal{W}^{i-1}$  of size at least  $\frac{1}{2}l_{i-1}$ , where

$$l_{i-1} = \frac{1}{2} \left( \ell(Q) - (s2^{-2i(n-1)}/i^{\lambda-(n-2)})^2 \right)$$

but when  $i = 2^j$ ,  $j \in \mathbb{N}$ , delete a rectangle of length  $s2^{-2j(n-1)}/2^{j(\lambda-(n-2))}$  of one edge and of length  $(s2^{-2j(n-1)}/2^{j(\lambda-(n-2))})^2$  of other edges from the centre of each cube in  $\mathcal{W}^{2^j-1}$  of size at least  $\frac{1}{2}l_{2^j-1}$ , where

$$l_{2^j-1} = \frac{1}{2} \left( \ell(Q) - \left( s2^{-2j(n-1)}/2^{j(\lambda-(n-2))} \right)^2 \right).$$

Then write

$$E = \left( \bigcap_{i \in M} \mathcal{W}^i \right) \cap \left( \bigcap_{j \in \mathbb{N}} \mathcal{W}^{2^j} \right),$$

which is our desired set. It is easy to check that the sum  $\sum_{i=1}^{\infty} \text{diam } Q_i \log^{\lambda-(n-2+\epsilon)}$  is finite for every  $\epsilon > 0$ , where  $Q_i$  are the complementary rectangles in Theorem 4.4. This completes the proof of the theorem.  $\square$

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